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LETTER TO THE EDITOR

On the metastable states of the zero-temperature SK mode

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Abstract. We generalize the calculation of the average number of metastable states of the Sherrington–Kirkpatrick spin-glass model at zero temperature to the case where there is an external magnetic field and the case where there is an asymmetric bond distribution. The results are in agreement with the existence of the AT instability at zero temperature for all finite values of external field and average bond value. We conclude the paper by examining the question of dual metastability, i.e. when spin configurations are metastable both with and without the presence of a magnetic field.

At zero temperature metastable states in the Sherrington–Kirkpatrick (SK) spin-glass model [11, 12] can be identified by those that are locally stable to single spin flips. At non-zero temperature the identification of metastable states is less obvious and most studies rely on the counting of the number of solutions to the celebrated TAP [14] equations. One of the indicators of the presence of a spin-glass phase is the appearance of an exponentially large (in the number of spins, N) number of solutions to the TAP equations as the system size goes to infinity. The resulting complicated free-energy landscape gives rise to the spin-glass phenomenon. The most general SK Hamiltonian is given by

$$H = -\sum_{ij}\left(J_{ij} + \frac{J_0}{N}\right)S_iS_j - h\sum_i S_i \tag{1}$$

where $\{S_i\}_{1\leqslant i\leqslant N}$ are Ising spins, J_{ij} are $N(0, 1/N)$ independent random variables (i.e. zero mean and variance $1/N$ Gaussians), J_0/N is the mean-bond value and h is the external magnetic field. For the regime $h = J_0 = 0$ the calculation of the number of solutions of the TAP equations has been carried out. The zero-temperature calculation is relatively straightforward [7, 3, 13] but the non-zero-temperature calculation is much more involved [2]; even so it has recently been extended to the general p-spin problem [10]. The calculation of the average number of metastable states of the Ising spin-glass chain at zero temperature has also been carried out in [6]. Expansions about the mean field for the number of metastable states in finite-range spin-glasses have also been investigated [3, 13].

It appears, however, that even the calculations at zero temperature for non-zero h have not been carried out; the case for asymmetric-bond distributions has been addressed in [3]. Of course, the general model has been extensively studied from the point of view of equilibrium statistical mechanics [9]. For example, the effects of non-zero h and J_0 on the AT line [4] indicating the onset of replica-symmetry breaking, are well known. In this letter we analyse the number of metastable states for non-zero h and then non-zero J_0. We

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are able to use these results to shed further light on the nature of the AT instability at zero temperature.

Recently there has been experimental and theoretical interest in how the application of an external uniform magnetic field affects the free-energy landscape of a spin-glass [15–17]. We therefore conclude this letter with a calculation of the average number of states in an SK spin-glass which are metastable both at zero field and in the presence of a uniform magnetic field.

The presence of an external magnetic field

In what follows we shall set $J_0 = 0$ but keep a non-zero external magnetic field h. The energy change ΔE_i involved in flipping the spin at site i is given by

$$\tfrac{1}{2}\Delta E_i = x_i = \sum_j J_{ij} S_i S_j + h S_i\,. \tag{2}$$

For a configuration to be stable to single spin flips we therefore require that $x_i > 0$ for all i. The joint probability density for the x_i is given by

$$p(\boldsymbol{x}) = \frac{1}{2^N}\mathrm{Tr}\left\langle \prod_{i=1}^{N} \delta\left(x_i - \sum_j J_{ij} S_i S_j - h S_i\right)\right\rangle \tag{3}$$

where the trace is over the spins and the angled brackets indicate averaging over disorder. The computation can be simplified using the standard gauge transformation $J_{ij} \to J_{ij} S_i S_j$ yielding

$$p(\boldsymbol{x}) = \frac{1}{2^N}\mathrm{Tr}\left\langle \prod_{i=1}^{N} \delta\left(x_i - \sum_j J_{ij} - h S_i\right)\right\rangle. \tag{4}$$

Averaging out the disorder, performing the trace over spins and making a Hubbard–Stratonovich transformation yields

$$p(\boldsymbol{x}) = \frac{N^{1/2}}{(2\pi)^{1/2}} \int \mathrm{d}z\, \mathrm{e}^{-Nz^2/2} \prod_{i=1}^{N} \frac{1}{(2\pi)^{1/2}} \exp\left(-\tfrac{1}{2}(x_i + z)^2 - \tfrac{1}{2}h^2\right) \cosh((x_i + z)h)\,. \tag{5}$$

The average number of metastable N_s is given by

$$N_s = 2^N \int_{\{x_i > 0\}} \mathrm{d}\boldsymbol{x}\, p(\boldsymbol{x}) = \frac{2^N N^{1/2}}{(2\pi)^{1/2}} \int \mathrm{d}z\, \exp\left(-N\left(\tfrac{1}{2}z^2 + \tfrac{1}{2}h^2 - \log(f(z,h))\right)\right) \tag{6}$$

where

$$f(z,h) = \frac{1}{(2\pi)^{1/2}} \int_{-z}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2} \cosh(hx)\,. \tag{7}$$

One should really calculate the average value of the log of the number of metastable states [3] and hence introduce replicas; indeed as pointed out in [3] the introduction of a uniform magnetic field should introduce strong correlations between the metastable states. However, we shall proceed with the calculation of N_s as it suffices to bring out the salient features of the problem.

In the limit of large N the integral over z can be carried out using the saddle-point method. The saddle-point equation is

$$z_c = \frac{\mathrm{e}^{-z_c^2/2} \cosh(h z_c)}{\int_{-z_c}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2} \cosh(hx)}\,. \tag{8}$$

We also have

$$\log(N_s) \sim NA(h) \tag{9}$$

where

$$A(h) = \log(2) - \tfrac{1}{2}\left(z_c^2 + h^2\right) + \log(f(z_c, h)) . \tag{10}$$

Equation (8) can be solved numerically and the graph of $A(h)$ versus h is shown in figure 1. For small values of h one finds that

$$A(h) = A(0) - \tfrac{1}{2}h^2 z_0^2 \tag{11}$$

where $z_0 \approx 0.506$, is the value of z_c obtained in the absence of an external magnetic field. For large values of h, one can show that

$$z_c \sim \left(\frac{2}{\pi}\right)^{1/2} e^{-h^2/2} \tag{12}$$

and hence

$$A(h) \sim \frac{1}{\pi} e^{-h^2} . \tag{13}$$

Consequently we can see that while the presence of a magnetic field does reduce the number of metastable states (as it introduces a tendency towards ferromagnetic behaviour), $A(h)$ still remains non-zero for arbitrarily large h and hence for any finite value of the external magnetic field the number of metastable states grows exponentially with N.

This result is in agreement with the observation that the AT instability occurs for all finite h in the SK model at zero temperature [4]. Therefore replica-symmetry breaking is employed to reflect the large number of metastable states present—this interplay is made more explicit in calculations based on the cavity method [8].

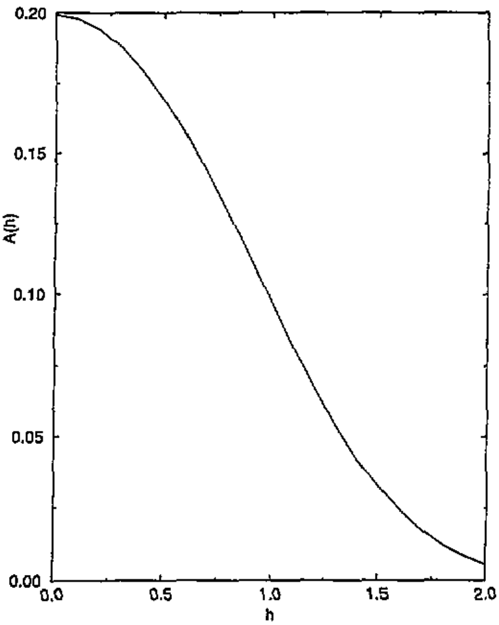

Figure 1. Graph of $A(h)$ versus h.

Asymmetric bond distribution

In what follows we shall set h to zero but introduce a non-zero value for J_0. In this case we find that the energy change ΔE_i due to flipping the spin at site i is given by

$$\tfrac{1}{2}\Delta E_i = x_i = \sum_j \left(J_{ij} + \frac{J_0}{N}\right) S_i S_j \,. \tag{14}$$

This scenario has been studied in [3], from a more general point of view, where the average of the log of the number of metastable states at fixed energy is calculated. Here we present the more straightforward calculation of the average number of metastable states, however, this allows a rather complete discussion of the various regimes involved. Proceeding as before we find

$$p(x) = \frac{1}{2^N}\mathrm{Tr}\left(\frac{1}{(2\pi)^{N+\frac{1}{2}}}\int \mathrm{d}z\, \mathrm{e}^{-z^2/2}\left(\prod_{i=1}^N \mathrm{d}\lambda_i\right)\right.$$
$$\left.\times \exp\left(\sum_i \lambda_i x_i - \tfrac{1}{2}\sum_i \lambda_i^2 + \mathrm{i}z\sum_i \lambda_i - \mathrm{i}\frac{J_0}{N}\sum_{ij}\lambda_i S_i S_j\right)\right). \tag{15}$$

At this stage we may make a second gauge transformation $\lambda_i \to \lambda_i S_i$. Defining

$$\bar{\lambda} = \frac{1}{N}\sum_i \lambda_i \tag{16}$$

and imposing the constraint via the Fourier representation of the delta function, and then tracing over the spins, we obtain

$$N_{\mathrm{s}} = \frac{2^N N^{3/2}}{(2\pi)^{3/2}}\int \mathrm{d}\mu\, \mathrm{d}\bar{\lambda}\, \mathrm{d}z\, \exp\left[N\left(-\tfrac{1}{2}z^2 - \tfrac{1}{2}\mu^2 + \mathrm{i}\bar{\lambda}\mu + \log(f(z,\mu,\bar{\lambda}))\right)\right] \tag{17}$$

where

$$f(z,\mu,\bar{\lambda}) = \frac{1}{(2\pi)^{1/2}}\int_{-z}^{\infty} \mathrm{d}x\, \mathrm{e}^{-x^2/2}\cosh(\mu x - \mathrm{i}\bar{\lambda}J_0)\,. \tag{18}$$

The resulting saddle-point equations are

$$z_{\mathrm{c}} = \frac{\mathrm{e}^{-z_{\mathrm{c}}^2/2}\cosh(\mu_{\mathrm{c}} z_{\mathrm{c}} + \lambda_{\mathrm{c}}^* J_0)}{\int_{-z_{\mathrm{c}}}^{\infty}\mathrm{d}x\, \mathrm{e}^{-x^2/2}\cosh(\mu_{\mathrm{c}} x + \lambda_{\mathrm{c}}^* J_0)} \tag{19}$$

$$\mu_{\mathrm{c}} = J_0 \frac{\int_{-z_{\mathrm{c}}}^{\infty}\mathrm{d}x\, \mathrm{e}^{-x^2/2}\sinh(\mu_{\mathrm{c}} x + \lambda_{\mathrm{c}}^* J_0)}{\int_{-z_{\mathrm{c}}}^{\infty}\mathrm{d}x\, \mathrm{e}^{-x^2/2}\cosh(\mu_{\mathrm{c}} x + \lambda_{\mathrm{c}}^* J_0)} \tag{20}$$

$$\lambda_{\mathrm{c}}^* = z_{\mathrm{c}}\tanh(\mu_{\mathrm{c}} z_{\mathrm{c}} + \lambda_{\mathrm{c}}^* J_0) \tag{21}$$

where $\lambda^* = -\mathrm{i}\bar{\lambda}$. In this case we have

$$\log(N_{\mathrm{s}}) \sim N A(J_0) \tag{22}$$

where

$$A(J_0) = \log(2) - \tfrac{1}{2}z_{\mathrm{c}}^2 - \tfrac{1}{2}\mu_{\mathrm{c}}^2 - \lambda_{\mathrm{c}}^*\mu + \log(f(z_{\mathrm{c}},\mu_{\mathrm{c}},\lambda_{\mathrm{c}}^*))\,. \tag{23}$$

The above equations clearly have the solution $\lambda_{\mathrm{c}}^* = \mu_{\mathrm{c}} = 0$ and $z_{\mathrm{c}} = z_0 \approx 0.506$ (i.e. the solution for z_{c} when $J_0 = 0$) giving $N_{\mathrm{s}} \sim A(0) \approx 0.2$; however, one can show that this solution is only a local maximum for $J_0 < 1/2z_0 \approx 0.988$. For $J_0 > 1/2z_0$ the value $A(J_0)$ decreases with increasing J_0, once again due to a tendency towards ferromagnetic ordering,

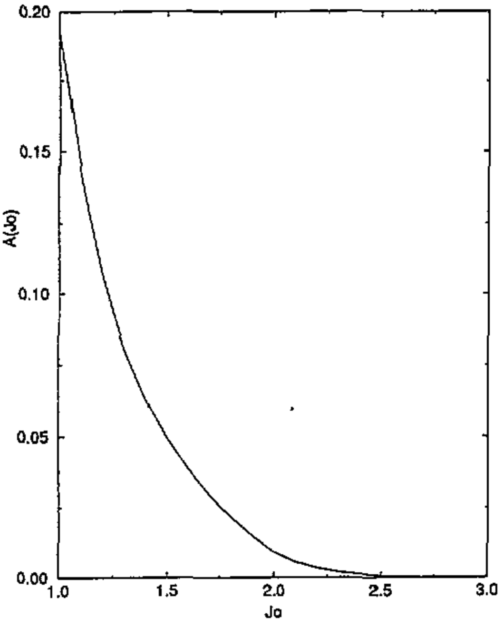

Figure 2. Graph of $A(J_0)$ versus J_0.

the graph of $A(J_0)$ versus J_0 for $J_0 \in [1/2z_0, 3]$ is shown in figure 2. For large positive values of J_0 one can show

$$z_c \sim \left(\frac{2}{\pi}\right)^{1/2} e^{-J_0^2/2} \qquad \mu_c \sim J_0 \qquad \lambda_c^* \sim \frac{2}{\pi} J_0 e^{-J_0^2} . \tag{24}$$

This leads to

$$A(J_0) \sim \frac{1}{\pi} e^{-J_0^2} . \tag{25}$$

Once again there exists an exponentially large number of metastable states for any finite J_0. For $J_0 > 1/2z_0$ the tendency towards ferromagnetic ordering reduces the value of A, whilst below this the introduction of non-zero J_0 has no effect on the result. Once again the result ties up with the fact that the AT instability is present at zero temperature for arbitrarily large J_0 [4].

Dual metastability

The effect of an external uniform magnetic field on the state space of a spin-glass has recently been considered in [15]. Experimentally one measures the AC susceptibility of the spin-glass for some time and then switches on the magnetic field. At constant field the susceptibility decays in a fashion which is well described by a power law and which depends to quite a high degree of accuracy only on ωt, where ω is the frequency of the external AC field [17]. These effects are well described by the trap model, corresponding to the random energy model, which was introduced in [1]. The effect of the uniform external magnetic field on the free-energy landscape, or equivalently the set of traps it is comprised of, is examined by observing the change in the susceptibility induced when a constant uniform magnetic field is applied. What is seen is a sudden increase in the susceptibility, it almost appears that the system has become younger. The observed effect can be interpreted in terms of a mixture of energy change in traps due to the Zeeman shift induced by the magnetic field and as traps being destroyed by the field. Really all of these effects are dynamical in nature and exhibit strong aging phenomena. However, an important aspect that one may tentatively examine is how traps may be destroyed by the magnetic field.

In the case of the SK model we may ask the question, what is the average number of states that are metastable both without and in the presence of a uniform magnetic field h, i.e. configurations which are dually metastable. One may then suggest that these states are related to the traps which are not destroyed by the magnetic field in the context of the model discussed in [15]. We shall denote this average *overlap* number by N_s^* and proceed by writing

$$N_s^* = \left\langle \mathrm{Tr} \prod_{i=1}^{N} \theta\left(\sum_j J_{ij} S_i S_j \right) \theta\left) \sum_j J_{ij} S_i S_j + h S_i \right) \right\rangle \tag{26}$$

where θ is the Heaviside step function. One proceeds in the same fashion as the previous calculations, using the Fourier representation of the θ's. After some straightforward algebra, one obtains

$$N_s^* = \frac{N^{1/2}}{(2\pi)^{1/2}} \int \mathrm{d}z \, \exp\left(-N\left(\tfrac{1}{2}z^2 - \log(f(z,h))\right)\right) \tag{27}$$

where

$$f(z,h) = \frac{1}{(2\pi)^{1/2}} \int_{-z}^{\infty} \mathrm{d}x \, \mathrm{e}^{-x^2/2}\left(1 + \exp\left(-|h|x - \tfrac{1}{2}h^2\right)\right). \tag{28}$$

In this case the saddle-point equation is

$$z_c = \frac{\mathrm{e}^{-z_c^2/2}(1 + \exp(|h|z_c - h^2/2))}{\int_{-z_c}^{\infty} \mathrm{d}x \, \mathrm{e}^{-x^2/2}(1 + \exp(-|h|x - h^2/2))}. \tag{29}$$

We thus obtain

$$\log(N_s^*) \sim N A^*(h) \tag{30}$$

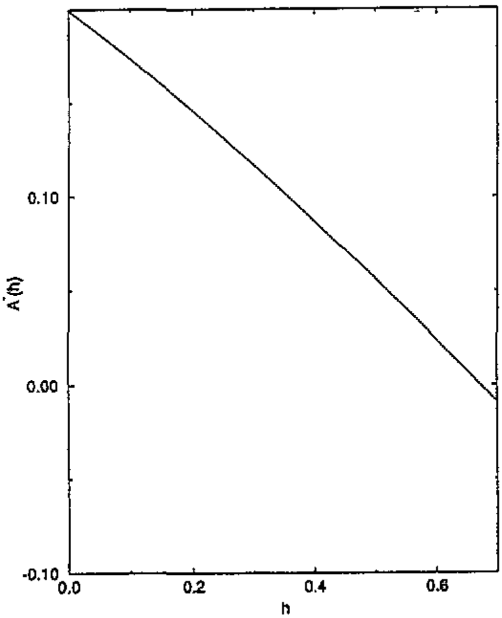

Figure 3. Graph of $A^*(h)$ versus h.

where

$$A^*(h) = -\tfrac{1}{2}z_c^2 + \log(f(z_c, h)) \,. \tag{31}$$

For small h one finds that

$$A^*(h) = A(0) - \tfrac{1}{2}|h|z_0 \,. \tag{32}$$

One immediately notices the difference in the small-field deviations from $A(0)$, between (11) and (32). It is clear intuitively that the double metastability constraint is harder to satisfy and therefore $A^*(h)$ should be less that $A(h)$, the change in the exponent is, however, more surprising. One can look at larger values of h numerically (see figure 3) and one finds that for a critical value of the external field $h_c \approx 0.671$, then $A^*(h)$ becomes negative and hence all the states that are metastable in the absence of the field are killed when it is applied. Whilst all of our previous results are to be expected from knowledge about the AT instability etc, this one is quite unforeseen.

There are many further questions one can ask in a similar vein, two important ones are (i) what is the overlap between metastable states in two non-zero but differing magnetic fields and (ii) what are the corresponding results for metastable states restricted to a fixed energy range? The latter question will have more applicability to experimental situations and both will be explored in a separate publication [5].

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